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# Investigation, Development, and Application of Optimal Output Feedback Theory

*Volume III—The Relationship Between Dynamic  
Compensators and Observers and Kalman Filters*

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## **FOREWORD**

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## ABSTRACT

This report investigates relationships between observers, Kalman Filters and dynamic compensators using feedforward control theory. In particular, the relationship, if any, between the dynamic compensator state and linear functions of a discrete plant state are investigated. It is shown that, in steady state, a dynamic compensator driven by the plant output can be expressed as the sum of two terms. The first term is a linear combination of the plant state. The second term depends on plant and measurement noise, and the plant control. Thus, the state of the dynamic compensator can be expressed as an estimator of the first term with additive error given by the second term. Conditions under which a dynamic compensator is a Kalman filter are presented, and reduced-order optimal estimators are investigated.

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## I. INTRODUCTION

In Appendix A of Reference 1, relationships were developed between dynamic compensators and observers. A basic question answered in Appendix A of Reference 1 is

- When is a dynamic compensator an observer of a linear function of plant states?

In this short report, the following more general question will be answered:

- What is the relationship between a dynamic compensator and the corresponding plant dynamics?

The dynamic compensator may or may not be an observer of plant states. This question is resolved using the theory of feedforward control developed in Reference 2. After the general question is answered, the solution is used to determine optimal Kalman filters and reduced-order optimal stochastic observers.

## II. FIRST REPRESENTATION

A compensator of any chosen order may be represented as

$$\underline{z}_{c,k+1} = \phi_c \underline{z}_{c,k} + \Gamma_c \underline{u}_{c,k} + \Gamma_u \underline{u}_k \quad (1)$$

The compensator state vector is  $\underline{z}_{c,k}$ , the compensator control vector is  $\underline{u}_{c,k}$  and  $\underline{u}_k$  is the plant control vector. The compensator is related to the plant by choosing  $\underline{u}_c$ , the compensator control, as follows

$$\underline{u}_{c,k} = \underline{y}_k \quad (2)$$

The plant is represented as

$$\underline{x}_{k+1} = \phi \underline{x}_k + \Gamma \underline{u}_k + \Gamma_w \underline{w}_k \quad (3)$$

$$\underline{y}_k = C \underline{x}_k + \underline{v}_k \quad (4)$$

The state vector is  $\underline{x}_k$ , the control vector is  $\underline{u}_k$ , as previously mentioned, and the white noise disturbance is  $\underline{w}_k$ . The measurements are  $\underline{y}_k$ . The white noise measurement disturbance vector is  $\underline{v}_k$ .

Using Reference 2 as a guide, the following equation is posed as the relationship between the plant and the compensator,

$$\begin{pmatrix} \underline{z}_{ck} \\ \underline{u}_{c,k} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \dots \\ S_{21} & S_{22} & S_{23} & \dots \end{pmatrix} \begin{pmatrix} \underline{x}_k \\ \underline{u}_k \\ \underline{u}_{k+1} \\ \underline{u}_{k+2} \\ \vdots \end{pmatrix} + \begin{pmatrix} A_{12} & A_{13} & \dots \\ A_{22} & A_{23} & \dots \end{pmatrix} \begin{pmatrix} \underline{w}_k \\ \underline{w}_{k+1} \\ \underline{w}_{k+2} \\ \vdots \end{pmatrix} \\ + \begin{pmatrix} B_{12} & B_{13} & \dots \\ B_{22} & B_{23} & \dots \end{pmatrix} \begin{pmatrix} \underline{v}_k \\ \underline{v}_{k+1} \\ \vdots \end{pmatrix} \quad (5)$$

The assumption about the relationship between the plant and compensator in Equation 5 becomes valid if all the unknown matrices in Equation 5 can be uniquely determined.

Substituting Equation 2 into Equation 5 yields,

$$S_{21} = C \quad (6)$$

$$S_{22} = S_{23} = \dots = 0 \quad (7)$$

$$A_{22} = A_{23} = \dots = 0 \quad (8)$$

$$B_{22} = I \quad (9)$$

$$B_{23} = B_{33} = \dots = 0 \quad (10)$$

From Equation 1, the following is true

$$\underline{z}_{c,k+1} - \underline{z}_{c,k} = [(\phi_c - I) \quad \Gamma_c] \begin{pmatrix} \underline{z}_{c,k} \\ \underline{u}_{c,k} \end{pmatrix} + \Gamma_u \underline{u}_k \quad (11)$$

From Equation 5, the following occurs

$$\begin{aligned} \underline{z}_{c,k+1} - \underline{z}_{c,k} = & [S_{11}(\phi - I) \quad S_{11}\Gamma - S_{12} \quad S_{12} - S_{13} \quad \dots] \begin{pmatrix} \underline{x}_k \\ \underline{u}_k \\ \underline{u}_{k+1} \\ \vdots \end{pmatrix} \\ & + [S_{11}\Gamma_w - A_{12} \quad A_{12} - A_{13} \quad \dots] \begin{pmatrix} \underline{w}_k \\ \underline{w}_{k+1} \\ \vdots \end{pmatrix} \\ & + [B_{12} \quad B_{12} - B_{13} \quad \dots] \begin{pmatrix} \underline{v}_k \\ \underline{v}_{k+1} \\ \vdots \end{pmatrix} \end{aligned} \quad (12)$$

Equating Equation 11 and Equation 12, results in the following algebraic relationships,

$$\begin{aligned} [(\phi_c - I)\Gamma_c] \begin{pmatrix} S_{11} & S_{12} & S_{13} & \dots \\ C & 0 & 0 & \dots \end{pmatrix} \\ = \begin{pmatrix} S_{11}(\phi - I) & S_{11}\Gamma - S_{12} - \Gamma_u & S_{12} - S_{13} & \dots \\ C & 0 & 0 & \dots \end{pmatrix} \end{aligned} \quad (13)$$

$$[(\phi_c - I)\Gamma_c] \begin{pmatrix} A_{12} & A_{13} & \dots \\ 0 & 0 & \dots \end{pmatrix} = \begin{pmatrix} S_{11}\Gamma_w - A_{12} & A_{12} - A_{13} & \dots \\ 0 & 0 & \dots \end{pmatrix} \quad (14)$$

$$[(\phi_c - I)\Gamma_c] \begin{pmatrix} B_{12} & B_{13} & \dots \\ I & 0 & \dots \end{pmatrix} = \begin{pmatrix} B_{12} & B_{12} - B_{13} & \dots \\ I & 0 & \dots \end{pmatrix} \quad (15)$$

From Equation 13, the  $S_{11}$  matrix satisfies the Lyapunov equation

$$\phi_c S_{11} + S_{11}(-\phi) = -\Gamma_c C \quad (16)$$

the other  $S$  matrices are obtained as

$$\phi_c S_{12} = S_{11} \Gamma - \Gamma_u \quad (17)$$

$$\phi_c S_{13} = S_{12} \quad (18)$$

$$\begin{aligned} \phi_c S_{14} &= S_{13} \\ &\vdots \end{aligned} \quad (19)$$

From Equation 14, the  $A$  matrices are obtained as

$$\phi_c A_{12} = S_{11} \Gamma_w \quad (20)$$

$$\begin{aligned} \phi_c A_{13} &= A_{12} \\ &\vdots \end{aligned} \quad (21)$$

From Equation 15, the  $B$  matrices are obtained as

$$\phi_c B_{12} = -\Gamma_c \quad (22)$$

$$\begin{aligned} \phi_c B_{13} &= B_{12} \\ &\vdots \end{aligned} \quad (23)$$

Using techniques developed in Reference 2, further simplifications of the relationships are possible. The compensator matrix,  $\phi_c$ , is assumed to be invertible. Substituting the solution of the  $S$ ,  $A$ , and  $B$  matrices back into Equation 5, and expressing Equation 5 in the  $z$  domain produces,



$$\begin{aligned}
\underline{z}_c = & [S_{11}(I + \phi_c^{-1}z + \phi_c^{-2}z^2 + \dots) \quad S_{12}] \begin{pmatrix} \underline{x} \\ \underline{u} \end{pmatrix} \\
& + [I + \phi_c^{-1}z + \phi_c^{-2}z^2 + \dots] A_{12} \underline{w} \\
& + [I + \phi_c^{-1}z + \phi_c^{-2}z^2 + \dots] B_{12} \underline{v}
\end{aligned} \tag{24}$$

$\underline{z}_c$ , the compensator state, and the z-domain variable,  $z$ , in Equation 24 should not be confused. It is easily shown that

$$(I - z\phi_c^{-1})^{-1} = I + \phi_c^{-1}z + \phi_c^{-2}z^2 + \dots \tag{25}$$

hence, define

$$\tilde{\underline{x}}_k = (I - z\phi_c^{-1})^{-1} [S_{12} \underline{u} + A_{12} \underline{w} + B_{12} \underline{v}] \tag{26}$$

Converting Equation 26 back to the time domain and using Equations 17, 20 and 22, the first important result is obtained,

$$\underline{z}_{c,k} = S_{11} \underline{x}_k + \tilde{\underline{x}}_k \tag{27}$$

$$\tilde{\underline{x}}_{k+1} = \phi_c \tilde{\underline{x}}_k - (S_{11} \Gamma - \Gamma_u) \underline{u}_k - S_{11} \Gamma_w \underline{w}_k + \Gamma_c \underline{v}_k \tag{28}$$

From Equation 28, it can be concluded that if the noise sources are zero, if  $\phi_c$  is stable and if,

$$\Gamma_u = S_{11} \Gamma \tag{29}$$

then the dynamic compensator is an observer and  $S_{11}$  determines what states in  $\underline{x}_k, \underline{z}_{c,k}$  observes, i.e.,

$$\underline{z}_{c,k} = S_{11} \underline{x}_k \quad (30)$$

If Equation 29 is not satisfied, then the dynamic compensator state observes the plant states plus an additional signal,  $\tilde{x}_k$ , caused by the plant controls.

The white noise assumption for  $\underline{w}_k$  and  $\underline{v}_k$  is not used in the derivation. The variables  $\underline{w}_k$  and  $\underline{v}_k$  show how any type of disturbance, i.e., biases, white noise, colored noise, etc., alter the dynamic compensator operation. It is rare in implementation that  $\Gamma_u$  can be chosen to equal  $S_{11} \Gamma$  exactly. Equations 27 and 28 are potentially useful in evaluating observer sensitivity to plant variations.

In the next section, the compensator structure is altered and rederived. The objective is to yield a Kalman filter implementation.

### III. SECOND REPRESENTATION-KALMAN FILTER APPROACH

The derivation in Section II is altered by assuming Equation 4 changes to

$$\underline{u}_{c,k} = \underline{y}_{k+1} \quad (31)$$

The dynamic compensator is now using the most recent measurement of  $\underline{y}$  to influence  $\underline{z}_c$ .

Equations 6 to 10 change to

$$S_{21} = C \phi \quad (32)$$

$$S_{22} = C \Gamma \quad (33)$$

$$S_{23} = S_{24} = \dots = 0 \quad (34)$$

$$A_{22} = C \Gamma_w \quad (35)$$

$$A_{23} = A_{24} = \dots = 0 \quad (36)$$

$$B_{22} = 0 \quad (37)$$

$$B_{23} = I \quad (38)$$

$$B_{24} = B_{25} = \dots = 0 \quad (39)$$

The other changes are

(Equation 16)

$$\phi_c S_{11} + S_{11}(-\phi) = -\Gamma_c C \phi \quad (40)$$

(Equation 17)

$$\phi_c S_{12} = S_{11} \Gamma - \Gamma_u - \Gamma_c C \Gamma \quad (41)$$

(Equation 20)

$$\phi_c A_{12} = S_{11} \Gamma_w - \Gamma_c C \Gamma_w \quad (42)$$

(Equation 22)

$$B_{12} = 0 \quad (43)$$

(Equation 23)

$$\phi_c B_{13} = -\Gamma_c \quad (44)$$

$$\phi_c B_{14} = B_{13} \quad (45)$$

The expression for  $\underline{z}_{c,k}$  in Equation 26 changes to

$$\underline{z}_{c,k} = S_{11} \underline{x}_k + \tilde{\underline{x}}_k \quad (46)$$

$$\begin{aligned} \tilde{\underline{x}}_{k+1} = & \phi_c \tilde{\underline{x}}_k - [(S_{11} - \Gamma_c C)\Gamma - \Gamma_u] \underline{u}_k \\ & - (S_{11} - \Gamma_c C)\Gamma_w \underline{w}_k + \Gamma_c \underline{v}_{k+1} \end{aligned} \quad (47)$$

In a Kalman filter formulation, the matrix  $S_{11}$  should be the identity matrix and  $\underline{u}_k$  should not affect the estimate. If these constraints are placed on Equation 46, Equation 40 changes to

$$\phi_c = (I - \Gamma_c C)\phi \quad (48)$$

$\Gamma_u$  is chosen to eliminate  $\underline{u}_k$  in  $\tilde{\underline{x}}_k$  in Equation 47, yielding

$$\Gamma_u = (I - \Gamma_c C)\Gamma \quad (49)$$

Substituting Equations 48 and 49 into Equation 3, yields,

$$\underline{z}_{c,k+1} = \phi \underline{z}_{c,k} + \Gamma \underline{u}_k + \Gamma_c \left( \underline{y}_{k+1} - C(\phi \underline{z}_{c,k} + \Gamma \underline{u}_k) \right) \quad (50)$$

Equation 50 is immediately recognizable as the update expression in a Kalman filter if  $\Gamma_c$  is a Kalman filter gain.

Substituting Equations 48 and 49 into Equations 46 and 47, yields,

$$\underline{z}_{c,k} = \underline{x}_k + \tilde{\underline{x}}_k \quad (51)$$

$$\tilde{\underline{x}}_{k+1} = (I - \Gamma_c C)\phi \tilde{\underline{x}}_k - (I - \Gamma_c C)\Gamma_w \underline{w}_k + \Gamma_c \underline{v}_{k+1} \quad (52)$$

where  $\tilde{\underline{x}}_k$  is the steady state noise corrupting signal to the estimate. The compensator gain  $\Gamma_c$  is the only unknown in Equations 51 and 52.

Clearly the objective of the compensator is to reduce the error term,  $\tilde{\underline{x}}_k$ , in Equation 51. The next section constructs an optimization problem for minimizing  $\tilde{\underline{x}}_k$ .

#### IV. OPTIMAL SOLUTIONS FOR COMPENSATOR DYNAMICS — THE KALMAN FILTER

One of the most important features of the results in Section III is that the compensator state is decomposed into a component related to the plant states and a component resulting from the noise sources in the plant and measurements. If there are elements in the compensator that are unknown, then one approach for choosing the unknown elements is to minimize the compensator state component caused by the noise sources.

As an example, consider the case discussed in Chapter III which resulted in a Kalman Filter structure. Equations 51 and 52 are

$$\underline{z}_{c,k} = \underline{x}_k + \tilde{\underline{x}}_k \quad (53)$$

$$\tilde{\underline{x}}_{k+1} = (I - \Gamma_c C)\phi \tilde{\underline{x}}_k - (I - \Gamma_c C)\Gamma_w \underline{w}_k + \Gamma_c \underline{v}_{k+1} \quad (54)$$

where  $\Gamma_c$ , the compensator gain, is an unspecified matrix. Assume that  $\Gamma_c$  is to be chosen to minimize the error,  $\tilde{\underline{x}}_k$ , in steady state, i.e.,

$$J = E \left\{ \tilde{\underline{x}}_k^T Q \tilde{\underline{x}}_k \right\} \quad (55)$$

The cost function can be rewritten as

$$J = \text{tr}\{Q P\} \quad (56)$$

where

$$P = E \left\{ \tilde{\underline{x}}_k \tilde{\underline{x}}_k^T \right\} \quad (57)$$

$P$  is the estimation error covariance in steady state. The steady-state estimation error covariance can be computed using Equation 54 to obtain,

$$P = (I - \Gamma_c C) \phi P \phi^T (I - C^T \Gamma_c^T) + (I - \Gamma_c C) W (I - C^T \Gamma_c^T) + \Gamma_c V \Gamma_c^T \quad (58)$$

$W$  is the covariance of the process noise,  $\Gamma_w \underline{w}_k$ , and  $V$  is the covariance of the measurement noise,  $\underline{v}_k$ . If the right hand side of Equation 58 is substituted into  $J$  and  $\partial J / \partial \Gamma_c$  is computed then,

$$0 = \frac{\partial J}{\partial \Gamma_c} = \Gamma_c (C \phi P \phi^T C^T + C W C^T + V) - \phi P \phi^T C^T - W C^T \quad (59)$$

Defining

$$P(-) = \phi P \phi^T + W \quad (60)$$

Then,  $\Gamma_c$  is given by

$$\Gamma_c = P(-) C^T [C P(-) C^T + V]^{-1} \quad (61)$$

The expression for  $\Gamma_c$  in Equation 61 is recognized as the steady-state optimal Kalman filter gain.  $P(-)$  in Equation 60, is the covariance of the error extrapolation as discussed in Reference 3.

It should be clear from the previous example that minimizing  $\tilde{x}_k$  causes the compensator to behave as an optimal estimator. The next objective of this report is to compute reduced-order optimal estimators. The next section derives the optimal  $(n - \ell)$  compensator that optimally observes the  $n - \ell$  plant states not measured in  $C$ .

## V. REDUCED-ORDER OPTIMAL ESTIMATORS

For reasons that will become clear later, the reduced-order optimal estimator is best designed assuming the following structure

$$\underline{z}_{c,k+1} = \phi_c \underline{z}_{c,k} + \Gamma_1 \underline{y}_{k+1} + \Gamma_2 \underline{y}_k + \Gamma_u \underline{u}_k \quad (62)$$

The relationship between the plant and the model in Equation 62 using the theory of feedforward control is

$$\underline{z}_{c,k} = S_{11} \underline{x}_k + \tilde{x}_k \quad (63)$$

$$\begin{aligned} \tilde{x}_{k+1} = & \phi_c \tilde{x}_k - ((S_{11} - \Gamma_1 C)\Gamma - \Gamma_u) \underline{u}_k \\ & - (S_{11} - \Gamma_1 C)\Gamma_w \underline{w}_k + \Gamma_1 \underline{y}_{k+1} + \Gamma_2 \underline{y}_k \end{aligned} \quad (64)$$

$$\phi_c S_{11} + S_{11} (-\phi) = -\Gamma_2 C - \Gamma_1 C \phi \quad (65)$$

Assume the plant states have been rearranged so that

$$\begin{pmatrix} \underline{y} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ 0 & I \end{pmatrix} \underline{x} \quad (66)$$

and  $C_1$  is an  $\ell \times \ell$  invertible matrix. The rearrangement can always be done if  $C$  is of full rank. Using the following transformation matrix to rearrange the plant states,

$$T = \begin{pmatrix} C_1 & C_2 \\ 0 & I \end{pmatrix} \quad ; \quad T^{-1} = \begin{pmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I \end{pmatrix} \quad (67 \text{ a,b})$$

results in the plant representation

$$\begin{pmatrix} \underline{y}_{k+1} \\ \underline{p}_{k+1} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \underline{y}_k \\ \underline{p}_k \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \underline{u}_k + \Gamma_{w1} \underline{w}_k \quad (68 \text{ a})$$

$$\underline{y}_k = [I \ 0] \begin{bmatrix} \underline{y}_k \\ \underline{p}_k \end{bmatrix} + \underline{v}_k \quad (68 \text{ b})$$

The objective is to find the optimal compensator which estimates only the  $(n - \ell)$   $\underline{p}$  states using only the  $\underline{y}$  measurements.

Consider applying the transformation

$$T_x = \begin{matrix} & \ell & n - \ell \\ \ell & & \\ n - \ell & \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} & \end{matrix} \quad (69)$$

to the plant. The matrix  $L$  is unknown. Multiplying the left side of Equation 68 by  $T_x$  in Equation 69 produces,

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} \underline{y}_{k+1} \\ \underline{p}_{k+1} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} + LA_{11} & A_{22} + LA_{12} \end{pmatrix} \begin{pmatrix} \underline{y}_k \\ \underline{p}_k \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 + LB_1 \end{pmatrix} \underline{u}_k + \Gamma_{w2} \underline{w}_k \quad (70)$$

The following equation for  $\underline{p}$  occurs as the lower partition in Equation 70,

$$\underline{p}_{k+1} = (A_{22} + LA_{12}) \underline{p}_k - L \underline{y}_{k+1} + (A_{21} + LA_{11}) \underline{y}_k + (B_2 + LB_1) \underline{u}_k + \Gamma_{w3} \underline{w}_k \quad (71)$$



The dynamic relationship for  $\underline{p}$  in Equation 71 is the equation which will be used for the compensator dynamics with the noise sources in Equation 71 eliminated. The need for  $\underline{y}_{k+1}$  and  $\underline{y}_k$  in the compensator equation is evident when Equation 71 is compared with Equation 62. The objective is to find the matrix  $L$  which minimizes  $\tilde{\underline{x}}$  where

$$\underline{z}_{c,k+1} = (A_{22} + L A_{12}) \underline{z}_{c,k} - L \underline{y}_{k+1} + (A_{21} + L A_{11}) \underline{y}_k + (B_2 + L B_1) \underline{u}_k \quad (72)$$

Comparing Equation 72 with Equation 62, the solution for  $S_{11}$  is

$$[A_{22} + L A_{12}] S_{11} + S_{11} \begin{pmatrix} -A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{pmatrix} = L[I \ 0] \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - [A_{21} + L A_{11}][I \ 0] \quad (73)$$

It is easily shown, by direct substitution, that

$$S_{11} = \begin{matrix} \ell & (n - \ell) \\ \ell[0 & I] \end{matrix} \quad (74)$$

satisfies Equation 73. The matrix  $S_{11}$  indicates that  $\underline{z}_{c,k}$ , using the structure in Equation 72, will observe  $\underline{p}_k$  as desired. The transformation matrix in Equation 69 is used in Reference 4 to construct the  $(n - \ell)$  reduced order observer. It can be shown that if  $(C, A)$  is an observable pair then  $(A_{12}, A_{22})$  is an observable pair. Observability guarantees that there exists a gain  $L$  which stabilizes the system  $(A_{12}, A_{22})$ .

From Equation 64, with  $\Gamma_u$  chosen to eliminate  $\underline{u}_k$ , i.e.,  $\Gamma_u = B_2 + L B_1$ , the estimation error dynamic equation reduces to

$$\tilde{\underline{x}}_{k+1} = (A_{22} + L A_{12}) \tilde{\underline{x}}_k - [L \ I] \Gamma_w \underline{w}_k - L \underline{v}_{k+1} + (A_{21} + L A_{11}) \underline{v}_k \quad (75)$$

Constructing the steady-state cost function yields,

$$J = E \left\{ \tilde{\underline{x}}_k^T Q \tilde{\underline{x}}_k \right\} = \text{tr}\{Q P\} \quad (76)$$

where

$$P = E \left\{ \tilde{\underline{x}}_k \tilde{\underline{x}}_k^T \right\} \quad (77)$$

$P$  is the steady state estimation error covariance. The optimal solution for the observer gain,  $L$ , which minimizes the cost function in Equation 77 when measurement noise corrupts the output,  $y_k$ , can be shown to be the solution of a cubic matrix equation. In this report, we will restrict attention to the case where the plant output contains no measurement noise. Using Equation 75, the steady state estimation error covariance satisfies the equation,

$$P = (A_{22} + L A_{12}) P (A_{22}^T + A_{12}^T L^T) + [I \ L] \overbrace{\begin{pmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{pmatrix}}^W \begin{pmatrix} I \\ L^T \end{pmatrix} \quad (78)$$

Substituting the right side of Equation 78 into Equation 76 and constructing  $\partial J / \partial L$ , results in

$$0 = \frac{\partial J}{\partial L} = L[A_{12} P A_{12}^T + W_3] + [A_{22} P A_{12}^T + W_2] \quad (79)$$

The solution for  $L$  is

$$L = -[A_{22} P A_{12}^T + W_2][A_{12} P A_{12}^T + W_3]^{-1} \quad (80)$$

The numerical solution for  $P$  and  $L$  can be obtained using Equations 78 and 80 and the approach in the next section. Note that the  $Q$  weighting matrix does not affect the  $L$  gain matrix.

## VI. OPTIMAL ESTIMATORS OF ARBITRARY ORDER

This section will describe a method for obtaining an optimal estimator of arbitrary order. Consider the plant shown in Equation 68. Assume the states have been reordered so that those states not measured in  $\underline{y}$  for which estimates are required are partitioned as the states  $\underline{p}_2$  in the following,

$$\begin{pmatrix} \underline{y}_{k+1} \\ \underline{p}_{1,k+1} \\ \underline{p}_{2,k+1} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \underline{y}_k \\ \underline{p}_{1,k} \\ \underline{p}_{2,k} \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \underline{u}_k + \Gamma_{w4} \underline{w}_k \quad (81)$$

The objective is to cause the dynamic compensator to optimally estimate only the  $\underline{p}_2$  states. Applying the following transformation to the plant

$$T = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ L & 0 & I \end{pmatrix} \quad (82)$$

yields the following equation for  $\underline{p}_2$

$$\begin{aligned} \underline{p}_{2,k+1} = & (A_{33} + L A_{13}) \underline{p}_{2,k} - L \underline{y}_{k+1} + (A_{31} + L A_{11}) \underline{y}_k \\ & + (B_3 + L B_1) \underline{u}_k + \Gamma_{w5} \underline{w}_k + (A_{32} + L A_{12}) \underline{p}_{1,k} \end{aligned} \quad (83)$$

A dynamic compensator can be constructed which observes  $\underline{p}_2$  if a matrix  $L$  can be found which stabilizes  $(A_{33} + L A_{13})$  and causes  $(A_{32} + L A_{12})$  to be a zero matrix.

The following cost function strives to achieve both objectives

$$J = E \left\{ \tilde{\underline{x}}_k^T Q_1 \tilde{\underline{x}}_k \right\} + \frac{1}{2} \text{tr} \left\{ (A_{32} + L A_{12}) Q_2 (A_{32} + L A_{12})^T \right\} \quad (84)$$

If  $L$  does not stabilize the compensator, then the estimation error,  $\tilde{\underline{x}}_k$ , would not be stable and  $J$  would not be finite. Increasing the  $Q_2$  weighting matrix causes  $L$  to minimize

$A_{32} + L A_{12}$ . The dynamic compensator is an observer of  $\underline{p}_{2,k}$  only if  $A_{32} + L A_{12}$  is zero. The form of the compensator is

$$\underline{z}_{c,k+1} = (A_{33} + L A_{13}) \underline{z}_{c,k} - L \underline{y}_{k+1} + (A_{31} + L A_{11}) \underline{y}_k + (B_3 + L B_1) \underline{u}_k \quad (85)$$

The solution for  $S_{11}$  is

$$[A_{33} + L A_{13}] S_{11} + S_{11} (-\phi) = L [I \ 0 \ 0] \phi - [A_{31} + L A_{11}] [I \ 0 \ 0] \quad (86)$$

If  $A_{32} + L A_{12}$  is not a zero matrix, then Equation 86 must be solved to determine exactly what combination of  $\underline{p}_1$  and  $\underline{p}_2$  the compensator state actually observes. The covariance equation for the estimation error without measurement noise is

$$P = (A_{33} + L A_{13}) P (A_{33}^T + A_{13}^T L^T) - (S_{11} + L [I \ 0 \ 0]) W (S_{11}^T + \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} L^T) \quad (87)$$

In order to minimize Equation 84, the equality constraints in Equations 86 and 87 are introduced into  $J$  using Lagrange multipliers,

$$\begin{aligned} J = & \text{tr} \{Q_1 P\} + \frac{1}{2} \text{tr} \{ (A_{32} + L A_{12}) Q_2 (A_{32} + L A_{12})^T \} \\ & + \{ [A_{33} + L A_{13}] S_{11} - S_{11} \phi - L [I \ 0 \ 0] \phi + [A_{31} + L A_{11}] [I \ 0 \ 0] \} X^T \\ & + \{ -P + (A_{33} + L A_{13}) P (A_{33}^T + A_{13}^T L^T) \\ & - (S_{11} + L [I \ 0 \ 0]) W (S_{11}^T + \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} L^T) \} Y^T \end{aligned} \quad (88)$$

The  $X$  and  $Y$  matrices are the Lagrange multipliers. The necessary conditions for a minimum for  $J$  are as follows:

$$0 = \frac{\partial J}{\partial \rho} = Q_1 - Y + (A_{33}^T + A_{13}^T L^T) Y (A_{33} + L A_{13}) \quad (89)$$

$$0 = \frac{\partial J}{\partial S_{11}} = [A_{33}^T + A_{13}^T L^T] X - X \phi^T + Y S_{11} W \quad (90)$$

$$\begin{aligned} 0 = \frac{\partial J}{\partial L} = & L(A_{12} Q_2 A_{12}^T) + 2Y L[A_{13} P A_{13}^T + W_1] + A_{32} Q_2 A_{12}^T \\ & + X S_{11}^T A_{13}^T - X \phi^T \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} + X \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} A_{11}^T \\ & + 2Y A_{33} P A_{13}^T \end{aligned} \quad (91)$$

An algorithm to solve for  $L$  based on the algorithm in Reference 5 would proceed as follows:

Choose an  $L$  which stabilizes  $A_{33} + L A_{13} = \phi_c$ .

Choose a scalar  $\alpha$  for which the following converges:

1. Solve for  $S_{11}$  in Equation 86.
2. Solve for  $Y$  in Equation 89.
3. Solve for  $X$  in Equation 90.
4. Solve for  $P$  in Equation 87.
5. Solve for  $L_{\text{new}}$  in Equation 91.
6. Choose  $L_{k+1} = L_k + \alpha(L_{\text{new}} - L_k)$ .
7. Check for convergence. If  $L_{k+1}$  destabilizes the plant, reduce  $\alpha$  until  $L_{k+1}$  stabilizes the plant. If convergence is not obtained, go to Step 1 and repeat.

Equation 91 has the following form

$$LA + BLC = D \quad (92)$$

If either  $A$  or  $B$  and  $C$  are invertible, Equation 92 can be rewritten as

$$L = \bar{A}L\bar{B} + \bar{C} \quad (93)$$

and solved using readily available algorithms for solving general Lyapunov equations. If  $A$  and  $B$  and  $C$  are not invertible, then  $L$  in Equation 92 can be solved using Kroneker products.

## VII. SUMMARY

The important contributions of this report are Equations 46 and 47 and Equations 63 and 64. Given a dynamic compensator of any order and a plant, these equations show how the compensator state is related to the plant states, controls, and noise sources. The conditions which cause the dynamic compensator to revert to an observer or Kalman filter are readily identified from the relationships in the equations. Optimal stochastic observers are developed by minimizing the estimation error identified in Equations 47 and 64.

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